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String spectrum of curved string backgrounds obtained by T-duality and shifts of polar angles

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Abstract: A class of exactly solvable string models can be obtained by starting with flat space and combining T-duality and shifts of angular coordinates of several polar planes. The models are the analog of the Lunin-Maldacena β-deformation of the $AdS_5 \times S^5$ type-IIB string background, which is dual to a Leigh-Strassler deformation of $\mathcal{N} = 4$ Super Yang-Mills Theory. We determine the complete physical string spectrum for two string models obtained in this way, by explicitly solving the string equations and quantizing in terms of free creation and annihilation operators. We also show that the 3-parameter (b_1, b_2, b_3) model, obtained by three independent TsT transformations, has tachyons in some regions of the parameter space.

Keywords: Conformal Field Models in String Theory, AdS-CFT and dS-CFT Correspondence.

Contents

1. Introduction

Conformal theories representing strings in curved backgrounds are in general extremely complicated and the physical string spectrum is known only in a few cases. One of these cases is the class of string backgrounds obtained by a sequence of T-duality transformations and shifts of periodic coordinates involving other periodic coordinates of different periods [1]. Due to the shifts, the new conformal field theories are not equivalent to the original flat-space starting point. By construction, the resulting model is nevertheless an exact conformal field theory, to all orders in the α' expansion. The models, being completely solvable, were used to test physical aspects of string propagation in curved spacetime, including closed superstrings in magnetic fields [2], supersymmetry breaking and closed string tachyons [2–4], D branes in magnetic fields [5], decay of type 0 string vacuum [6, 7], spacetime singularities [1], strings in plane wave backgrounds [8, 9] and closed time-like curves [10].

Finding new conformal σ models of strings in curved backgrounds where the string spectrum can be found exactly is of obvious interest. New solvable models have been recently introduced by Lunin and Maldacena in [11]. The models are constructed by applying a T duality transformation, then a shift of a periodic coordinate, and then another T-duality transformation. These transformations were generally referred as TsT transformations in [12], and here we will adopt this name. In contrast to [1], where the shift involves an S^1 coordinate, in [11] the shift only involves polar angles. This novel application of TsT transformations gives rise to new exactly solvable string models which have not been studied before.

The main motivation for the study of these models is that they are closely related to the analogous deformation of $AdS_5 \times S^5$ that yields a background $(AdS_5 \times S^5)_\beta$, which is the supergravity dual of the Leigh-Strassler β-deformation of $\mathcal{N} = 4$ Super Yang-Mills

Theory to a $\mathcal{N}=1$ superconformal gauge theory [13] (some recent interesting works on the $(AdS_5\times S^5)_\beta$ string theory can be found in [14−25]). The Lunin-Maldacena deformation applies to the S^5 part of the space. The 5-sphere can be represented by three complex planes with the restriction $z_1z_1^*+z_2z_2^*+z_3z_3^*=R^2$, where R is the radius of the sphere (related to the 't Hooft coupling $\lambda=g^2_{\rm YM}N$ by $R^2=\sqrt{\lambda}\,\alpha'$). The present models are obtained by a similar deformation applied on a flat space (z_1,z_2,z_3).

This paper is organized as follows. In section 2 we consider the simplest model involving two complex planes, and find the mass spectrum of quantum superstring states. A model involving three complex planes related to the Lunin-Maldacena deformation is then considered in section 3.1. In section 3.2 we consider a model which is the analog of the three-parameter deformation of $AdS_5\times S^5$ introduced by Frolov [15] and further studied in [23]. We find tachyon states in some regions of the parameter space. Finally, section 4 contains some remarks on the string spectrum in β-deformed AdS backgrounds.

2. String model I: TsT on two polar planes

This model was introduced in [11] to illustrate TsT transformations in a simple setting. The starting lagrangian is

$$L=\partial_+x_\mu\partial_-x^\mu+\partial_+r_1\partial_-r_1+r_1^2\partial_+\varphi_1\partial_-\varphi_1+\partial_+r_2\partial_-r_2+r_2^2\partial_+\varphi_2\partial_-\varphi_2\,,\qquad \mu=0,1,\ldots,5\,. \tag{2.1}$$

We use the notation $\sigma_\pm=\tau\pm\sigma$. For simplicity in the presentation, here we have written bosonic fields only. Restoring fermion contributions in the formulas is straightforward and will be done at the end. In what follows we will omit from most formulas the contribution of the free coordinates x^μ, which are decoupled and can be treated as in standard free superstring theory. After T-duality in the φ_1 coordinate to a new coordinate $\tilde\varphi_1$, and a shift $\varphi_2\to\varphi_2+b\tilde\varphi_1$, the lagrangian becomes

$$\begin{aligned}L&=\partial_+r_1\partial_-r_1+r_1^{-2}\partial_+\tilde\varphi_1\partial_-\tilde\varphi_1+\partial_+r_2\partial_-r_2+r_2^2(\partial_+\varphi_2+b\partial_+\tilde\varphi_1)(\partial_-\varphi_2+b\partial_-\tilde\varphi_1)+\\&\quad+\mathcal{R}\left(\phi_0-\frac{1}{2}\log r_1^2\right),\end{aligned} \tag{2.2}$$

where

$$\mathcal{R}=\frac{1}{4}\alpha'\sqrt{g}R^{(2)}\,,$$

and $\tilde\varphi_1$ has period $2\pi\alpha'$. Finally, by performing a T-duality back in $\tilde\varphi_1$, one gets the lagrangian

$$\begin{aligned}L&=\partial_+r_1\partial_-r_1+\partial_+r_2\partial_-r_2+F(r_1^2\partial_+\varphi_1\partial_-\varphi_1+r_2^2\partial_+\varphi_2\partial_-\varphi_2)+\\&\quad+bF\,r_1^2r_2^2(\partial_+\varphi_2\partial_-\varphi_1-\partial_+\varphi_1\partial_-\varphi_2)+\mathcal{R}\left(\phi_0+\frac{1}{2}\log F\right),\\F&\equiv(1+b^2r_1^2r_2^2)^{-1}\,.\end{aligned} \tag{2.3}$$

This describes strings propagating in the supergravity background

$$\begin{aligned}ds^2&=dr_1^2+dr_2^2+F(r_1^2d\varphi_1^2+r_2^2d\varphi_2^2)\,,\\B_{12}&=b\,r_1^2r_2^2F\,,\qquad e^{2\phi}=e^{2\phi_0}F\,.\end{aligned} \tag{2.4}$$

By construction, this background is a solution of the string equations to all orders in α'. The reason is that the model is equivalent to (2.2) as a CFT, being related by T-duality, and (2.2) is locally equivalent to a background related to flat space by T-duality.

In order to determine the physical string spectrum, one can either consider the lagrangian (2.2) or (2.3), since they are equivalent as CFT. We shall follow closely ref. [1], section 5, where a class of string models obtained by T-duality and shifts are solved, since there are similarities in the structure of the solution.

In general, the solution to the string equations of motion for two T-dual σ models are related by $(G_{\mu\nu} + B_{\mu\nu})\partial_\pm x^\nu = \mp\partial_\pm \tilde{x}_\mu$. Using this relation, we find the general solution to the string equations of motion in the curved background (2.4),

$$\varphi_1 = \frac{1}{2i}\log\frac{X_1}{X_1^*} + b\tilde{\varphi}_2\,, \qquad \varphi_2 = \frac{1}{2i}\log\frac{X_2}{X_2^*} - b\tilde{\varphi}_1\,, \tag{2.5}$$

where

$$\begin{aligned} X_1 = X_{1+}(\sigma_+) + X_{1-}(\sigma_-)\,, \quad X_2 = X_{2+}(\sigma_+) + X_{2-}(\sigma_-)\,, \\ \partial_\pm\tilde{\varphi}_i = \pm\frac{i}{2}\Big(X_i^*\partial_\pm X_i - X_i\partial_\pm X_i^*\Big)\,. \end{aligned} \tag{2.6}$$

Hence

$$\begin{aligned} \tilde{\varphi}_i = 2\pi\alpha'\Big(J_{i-}(\sigma_-) - J_{i+}(\sigma_+)\Big) + \frac{i}{2}(X_{i+}X_{i-}^* - X_{i-}X_{i+}^*)\,, \qquad i = 1,2\,, \\ J_{i\pm}(\sigma_\pm) \equiv \frac{i}{4\pi\alpha'}\int_0^{\sigma_\pm} d\sigma_\pm(X_{i\pm}\partial_\pm X_{i\pm}^* - X_{i\pm}^*\partial_\pm X_{i\pm})\,. \end{aligned} \tag{2.7}$$

Using

$$\begin{aligned} \varphi_i(\sigma+\pi,\tau) &= \varphi_i(\sigma,\tau) + 2\pi n\,, \\ \tilde{\varphi}_i(\sigma+\pi,\tau) &= \tilde{\varphi}_i(\sigma,\tau) - 2\pi\alpha' J_i\,, \qquad dJ = J_{iL} + J_{iR}\,, \\ J_{iL} &= J_{i+}(\pi)\,, \qquad J_R = J_{i-}(\pi)\,, \end{aligned} \tag{2.8}$$

one finds that the free fields $X_1\,,\ X_2$ satisfy the twisted boundary conditions

$$X_1(\sigma+\pi,\tau) = e^{2\pi i\nu_1}X_1(\sigma,\tau)\,, \qquad X_2(\sigma+\pi,\tau) = e^{2\pi i\nu_2}X_2(\sigma,\tau)\,, \tag{2.9}$$

with

$$\nu_1 = \alpha' b J_2\,, \qquad \nu_2 = -\alpha' b J_1\,. \tag{2.10}$$

Note that ν_1, ν_2 are defined modulo n, $n =$ integer. These boundary conditions are satisfied by writing

$$X_{i\pm} = e^{\pm 2i\nu_i\sigma_\pm}\chi_{i\pm}\,, \qquad \chi_i(\sigma+\pi,\tau) = \chi_i(\sigma,\tau)\,. \tag{2.11}$$

The fields $\chi_{i\pm}$ are single-valued and can be expanded as follows

$$\chi_{i-} = i\sqrt{\frac{\alpha'}{2}}\sum_n a_{ni}e^{-2in\sigma_-}\,, \qquad \chi_{i+} = i\sqrt{\frac{\alpha'}{2}}\sum_n \tilde{a}_{ni}e^{-2in\sigma_+}\,. \tag{2.12}$$

In terms of the free fields $X_1,\ X_2$, the energy-momentum tensor of the string model (2.3) is simply

$$T_{\pm\pm} = \partial_\pm X_1 \partial_\pm X_1^* + \partial_\pm X_2 \partial_\pm X_2^* \,. \tag{2.13}$$

This can be checked by plugging (see eq. (2.5))

$$X_i = r_i e^{i\phi_i}\,, \qquad \phi_i = \varphi_i - b\epsilon_{ij}\tilde{\varphi}_j \,. \tag{2.14}$$

In this way we recover the $T_{\pm\pm}$ in terms of $\varphi_1,\ \varphi_2$ that follows directly from the original lagrangian (2.3),

$$T_{\pm\pm} = \sum_{i=1}^{2} \left[\partial_\pm r_i \partial_\pm r_i + \frac{r_i^2}{1+b^2 r_1^2 r_2^2} \partial_\pm \varphi_i \partial_\pm \varphi_i \right] . \tag{2.15}$$

Inserting (2.11) in (2.13), the energy-momentum tensor components $T_{\pm\pm}$ take the form

$$T_{++} = \sum_{i=1}^{2} \left[\partial_+ \chi_{i+} \partial_+ \chi_{i+}^* + 2i\nu_i (\chi_{i+} \partial_+ \chi_{i+}^* - \chi_{i+}^* \partial_+ \chi_{i+}) + 4\nu_i^2 \chi_{i+}^* \chi_{i+} \right] , \tag{2.16}$$

$$T_{--} = \sum_{i=1}^{2} \left[\partial_- \chi_{i-} \partial_- \chi_{i-}^* - 2i\nu_i (\chi_{i-} \partial_- \chi_{i-}^* - \chi_{i-}^* \partial_- \chi_{i-}) + 4\nu_i^2 \chi_{i-}^* \chi_{i-} \right] . \tag{2.17}$$

Inserting the expansions (2.12) and integrating over σ, we find the Virasoro operators

$$L_0 = \frac{1}{2} \sum_{i=1}^{2} \sum_n (n+\nu_i)^2 a_{ni}^* a_{ni}\,, \qquad \tilde{L}_0 = \frac{1}{2} \sum_{i=1}^{2} \sum_n (n-\nu_i)^2 \tilde{a}_{ni}^* \tilde{a}_{ni}\,. \tag{2.18}$$

We will also need the expression of the angular momentum components which are conjugate to ϕ_1 and ϕ_2. In terms of the mode operators, they are given by

$$J_{iR} = -\frac{1}{2} \sum_n (n+\nu_i) a_{ni}^* a_{ni}\,, \qquad J_{iL} = -\frac{1}{2} \sum_n (n-\nu_i) \tilde{a}_{ni}^* \tilde{a}_{ni}\,. \tag{2.19}$$

Let us now consider the operator quantization of the model. Canonical commutation relations for $x_i \equiv r_i e^{i\varphi_i}$ imply

$$[P_{X_i}(\sigma,\tau), X_j^*(\sigma',\tau)] = -i\delta_{ij}\delta(\sigma-\sigma')\,. \tag{2.20}$$

Hence

$$[a_{ni}, a_{mj}^*] = 2(n+\nu_i)^{-1}\delta_{ij}\delta_{nm}\,, \qquad [\tilde{a}_{ni}, \tilde{a}_{mj}^*] = 2(n-\nu_i)^{-1}\delta_{ij}\delta_{nm}\,. \tag{2.21}$$

We now introduce standard creation and annihilation operators $b_{n\pm},\ b_{n\pm}^\dagger$, satisfyng $[b, b^\dagger] = 1$ by a proper rescaling of $a_n,\ \tilde{a}_n$ as in [1],

$$\begin{aligned}
b_{n+} &= a_{-n}^* \omega_-\,, \qquad b_{n-} = a_n \omega_+\,, \\
\tilde{b}_{n+} &= \tilde{a}_{-n}^* \omega_+\,, \qquad \tilde{b}_{n-} = \tilde{a}_n \omega_-\,, \qquad b_0 = \sqrt{\frac{\nu}{2}}\, a_0\,, \qquad \tilde{b}_0 = \sqrt{\frac{\nu}{2}}\, \tilde{a}_0^*\,, \\
\omega_\pm &\equiv \sqrt{\frac{1}{2}(n\pm\nu)}\,, \qquad n = 1,2,\ldots\,, \qquad 0<\nu<1\,,
\end{aligned} \tag{2.22}$$

where indices 1 and 2 have been omitted. The Virasoro operators then take the form

$$
\begin{aligned}
L_0 &= \frac{1}{4}\alpha' p_\mu^2 + (\hat{N}_R - \nu_1 \hat{J}_{1R} - \nu_2 \hat{J}_{2R})\,, \\
\tilde{L}_0 &= \frac{1}{4}\alpha' p_\mu^2 + (\hat{N}_L + \nu_1 \hat{J}_{1L} + \nu_2 \hat{J}_{2L})\,,
\end{aligned} \tag{2.23}
$$

where $\hat{J}_{iR}$, $\hat{J}_{iL}$ are given by

$$
\hat{J}_R = J_R - \frac{1}{2} = -b_0^\dagger b_0 - \frac{1}{2} + \sum_{n=1}^{\infty}(b_{n+}^\dagger b_{n+} - b_{n-}^\dagger b_{n-}) + K_R\,, \tag{2.24}
$$

$$
\hat{J}_L = J_L + \frac{1}{2} = \tilde{b}_0^\dagger \tilde{b}_0 + \frac{1}{2} + \sum_{n=1}^{\infty}(\tilde{b}_{n+}^\dagger \tilde{b}_{n+} - \tilde{b}_{n-}^\dagger \tilde{b}_{n-}) + K_L\,, \tag{2.25}
$$

$$
K_R^{(\mathrm{NS})} = -\sum_{r=1/2}^{\infty}(c_r^* c_r + c_{-r} c_{-r}^*)\,, \qquad K_R^{(\mathrm{R})} = -[d_0^*, d_0] + \sum_{n=1}^{\infty}(d_n^* d_n + d_{-n} d_{-n}^*)\,, \tag{2.26}
$$

and there is an index $i = 1, 2$ in all mode operators that has been omitted for the sake of clarity. The expression for K_L is similar, with tildes in the mode operators. We have restored the fermion contributions, following the notation of [2] (c_r and d_n are the fermion mode operators in the NS and R sector, respectively). The eigenvalues of $\hat{J}_{L,R}$ are

$$
\hat{J}_{L,R} = \pm\left(l_{L,R} + \frac{1}{2}\right) + S_{L,R}\,, \qquad \hat{J} = \hat{J}_L + \hat{J}_R = l_L - l_R + S_L + S_R\,, \tag{2.27}
$$

where l_L, $l_R = 0, 1, 2, \ldots$ are Landau quantum numbers and the spin $S_L + S_R$ is an integer in the NS-NS and R-R sectors, and half-integer in the NS-R and R-NS sectors.

The operators $\hat{N}_R$, $\hat{N}_L$ have the standard expression in terms of free creation and annihilation operators, $\hat{N}_{R,L} = N_{R,L} - a$, with $a^{(\mathrm{NS})} = 1/2$, $a^{(\mathrm{R})} = 0$, where e.g. in the Ramond sector,

$$
N_R = \sum_{n=1}^{\infty} n\left(b_{ni+}^\dagger b_{ni+} + b_{ni-}^\dagger b_{ni-} + a_{n\alpha}^\dagger a_{n\alpha} + d_{ni}^* d_{ni} + d_{-ni} d_{-ni}^* + d_{-n\alpha} d_{n\alpha}\right), \tag{2.28}
$$

where summation over i is understood and $a_{n\alpha}$, $d_{n\alpha}$ stand for the remaining ($\alpha = 1, \ldots, 4$) transverse mode operators. For physical states satisfying the GSO condition, the eigenvalues are $\hat{N}_{L,R} = 0, 1, 2, \ldots$. The hamiltonian and level matching constraints are

$$
L_0 + \tilde{L}_0 = 0\,, \qquad L_0 = \tilde{L}_0\,. \tag{2.29}
$$

They lead to the string spectrum:

$$
\alpha' M^2 = 2(\hat{N}_R - \hat{\nu}_1 \hat{J}_{1R} - \hat{\nu}_2 \hat{J}_{2R}) + 2(\hat{N}_L + \hat{\nu}_1 \hat{J}_{1L} + \hat{\nu}_2 \hat{J}_{2L})\,, \tag{2.30}
$$

$$
\hat{N}_R = \hat{N}_L\,, \tag{2.31}
$$

where $\hat{\nu}_i = \nu_i - [\nu_i]$, and

$$
\nu_1 = \alpha' b(\hat{J}_{2R} + \hat{J}_{2L})\,, \qquad \nu_2 = -\alpha' b(\hat{J}_{1R} + \hat{J}_{1L})\,. \tag{2.32}
$$

Note that $\alpha' b$ is dimensionless.

A few remarks are in order:

- The spectrum has a periodic dependence on the twist parameters ν_i. This must be the case, since the boundary conditions (2.11) are unchanged if we replace $\nu_i \to \nu_i + n_i$, $n_i =$ integer. Due to the presence of fermions, the actual periodicity is $\nu_i \to \nu_i + 2n_i$. When $2n_i \le \nu_i < 2n_i + 1$, $i = 1, 2$ one should use the standard GSO projection. When one of the ν_i is in the interval $2n_i + 1 \le \nu_i < 2n_i + 2$, one should use the reversed GSO projection, meaning that only states having half-integer eigenvalues of the operators $\hat{N}_{L,R}$ will survive, i.e. $\hat{N}_{L,R} = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots$ (see [2]).

- The fact that in (2.30) $\hat{\nu}_i \hat{J}_{iL}$ appear with the opposite sign of $\hat{\nu}_i \hat{J}_{iR}$ is due to our conventions. What is independent of conventions is that the terms proportional to the Landau numbers l_{iL} and l_{iR} both contribute with positive sign to M^2, and this is of course the case in eq. (2.30).

- If $\alpha' b$ is irrational, then ν_1, ν_2 are not integer numbers for any $\hat{J}_1, \hat{J}_2 \neq 0$. When $\alpha' b$ is rational, $\alpha' b = p/q$, there are sectors $\hat{J}_1$ or $\hat{J}_2 = qn$, where one of the ν_i is an integer and the corresponding $\hat{\nu}_i$ vanishes.

- If one of the $\hat{\nu}_i$ vanishes, say $\hat{\nu}_2$, then the zero mode structure in the plane 2 changes. The oscillator modes $b_{02}, b_{02}^\dagger$ and $\tilde{b}_{02}, \tilde{b}_{02}^\dagger$ (giving rise to Landau numbers l_{2R}, l_{2L}) are replaced by x_2, x_2^*, p_2, p_2^*.

- If $\hat{\nu}_1$, $\hat{\nu}_2 \neq 0$, there are no tachyons for any b (see below, and also section 3.2, where we give a discussion in a similar model).

As an application, let us consider states of minimal energy for a given level $N \equiv \hat{N}_R = \hat{N}_L$. From the explicit representation in terms of creation and annihilation operators, one can see that the S_{iR}, S_{iL} satisfy the bounds

$$|S_{1R} \pm S_{2R}| \le \hat{N}_R + 1 \,, \qquad |S_{1L} \pm S_{2L}| \le \hat{N}_L + 1 \,. \tag{2.33}$$

We consider a state having $l_{iL} = l_{iR} = 0$, and

$$S_{1R} + S_{2R} = N + 1 \,, \qquad S_{1L} + S_{2L} = -N - 1 \,,$$

It follows that $\hat{J}_1 = -\hat{J}_2$. We assume $\hat{J}_2 = S_{2L} + S_{2R} > 0$ and $0 < \alpha' b < 1$. Then $\nu_1 = \nu_2 = \alpha' b\, s$, $s \equiv S_{2L} + S_{2R}$. The mass formula takes the form

$$\alpha' M^2 = 4N(1 - (\nu_1 - [\nu_1])) \,. \tag{2.34}$$

This is manifestly positive definite. It is possible to take a limit, which is similar to the limits studied in [26], where most string states decouple. The number of surviving states at each level is proportional to N. In the present case, we consider $\alpha' b = 1 - \varepsilon$. Then

$$\nu_1 - [\nu_1] = \alpha' b\, s - [\alpha' b\, s] = 1 - \varepsilon s \,,$$

where we have assumed $\varepsilon s < 1$, which is a valid assumption for any given s, since we are going to take the limit $\varepsilon \to 0$. Now write $\alpha' = \varepsilon \alpha'_{\rm eff}$, and take the limit $\varepsilon \to 0$ with fixed

$\alpha'_{\rm eff}$. In this limit, the masses of all states with $S_{1R}+S_{2R}<N+1$ or $S_{1L}+S_{2L}>-N-1$ go to infinity. For the special states considered above, the mass formula (2.34) takes the form

$$\alpha'_{\rm eff}M^2=4N\,s\,,\qquad 0<s\leq 2N+2\,, \tag{2.35}$$

Thus these states have finite mass after the limit $\varepsilon\to 0$. Note that one can also consider a limit with $\alpha' b=p/q-\varepsilon$, where there are surviving states.

3. String model II: TsT on three polar planes

3.1 One-parameter deformation

The starting point is the string theory lagrangian

$$L=\sum_{i=1}^{3}\left(\partial_+ r_i\partial_- r_i+r_i^2\partial_+\phi_i\partial_-\phi_i\right), \tag{3.1}$$

or, in cartesian coordinates,

$$L=\sum_{i=1}^{3}\partial_+ X_i\partial_- X_i^*\,, \tag{3.2}$$

where

$$\phi_1=\psi-\varphi_1'\,,\qquad \phi_2=\psi-\varphi_2'\,,\qquad \phi_3=\psi+\varphi_1'+\varphi_2'\,. \tag{3.3}$$

Here we omit other free coordinates in the string theory lagrangian as well as fermion contributions. They will be incorporated later.

Now we proceed as in the model of section 2, by performing a T-duality transformation in the φ_1' variable to a new variable $\tilde\varphi_1$, and a shift: $\varphi_2'=\varphi_2+b\tilde\varphi_1$. After T-duality in $\tilde\varphi_1$ to the T-dual variable φ_1, one obtains a final lagrangian which is symmetric in φ_1, φ_2, representing a curved string background with B-field components and dilaton. This model is constructed in the appendix A of [11].

Using the relation $(G_{\mu\nu}+B_{\mu\nu})\partial_\pm x^\nu=\mp\partial_\pm\tilde x_\mu$ between the solutions to the string equations of motion for two T-dual σ models, we find the solution

$$\begin{aligned}\varphi_1&=\varphi_1'+b\tilde\varphi_2=\frac{1}{3}(\phi_2+\phi_3-2\phi_1)+b\tilde\varphi_2\,,\\ \varphi_2&=\varphi_2'-b\tilde\varphi_1=\frac{1}{3}(\phi_1+\phi_3-2\phi_2)-b\tilde\varphi_1\,,\\ \psi&=\frac{1}{3}(\phi_1+\phi_2+\phi_3)\,,\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}\partial_\pm\tilde\varphi_1&=\mp\left(r_3^2\partial_\pm\phi_3-r_1^2\partial_\pm\phi_1\right),\\ \partial_\pm\tilde\varphi_2&=\mp\left(r_3^2\partial_\pm\phi_3-r_2^2\partial_\pm\phi_2\right).\end{aligned} \tag{3.5}$$

Using (3.3), (3.4) and the fact that φ_1, φ_2 and ψ are 2π periodic, we find the boundary conditions of ϕ_i variables,

$$\begin{aligned} \phi_1(\sigma+\pi) &= \phi_1(\sigma) + b\Delta\tilde{\varphi}_2\,, \\ \phi_2(\sigma+\pi) &= \phi_2(\sigma) - b\Delta\tilde{\varphi}_1\,, \\ \phi_3(\sigma+\pi) &= \phi_3(\sigma) - b\Delta\tilde{\varphi}_2 + b\Delta\tilde{\varphi}_1\,, \end{aligned} \tag{3.6}$$

with

$$\Delta\tilde{\varphi}_1 = 2\pi\alpha'(J_1 - J_3)\,, \qquad \Delta\tilde{\varphi}_2 = 2\pi\alpha'(J_2 - J_3)\,, \tag{3.7}$$
$$J_i = J_{iL} + J_{iR}\,, \qquad J_{iL} = J_{i+}(\pi)\,, \qquad J_{iR} = J_{i-}(\pi)\,.$$

The operators J_i are as in (2.7), with $i = 1, 2, 3$.

Thus the twists ν_i (defined as in (2.9)) in the free fields X_i are given by

$$\nu_1 = \alpha' b(J_2 - J_3)\,, \qquad \nu_2 = \alpha' b(J_3 - J_1)\,, \qquad \nu_3 = \alpha' b(J_1 - J_2)\,. \tag{3.8}$$

We then proceed exactly as in section 2: we redefine X_i in terms of single-valued fields χ_i as in (2.11), and the the expressions that follow are the same as in section 2, with the only difference that now $i = 1, 2, 3$.

Therefore, we find the string spectrum

$$\alpha' M^2 = 2(\hat{N}_R - \hat{\nu}_1 \hat{J}_{1R} - \hat{\nu}_2 \hat{J}_{2R} - \hat{\nu}_3 \hat{J}_{3R}) + 2(\hat{N}_L + \hat{\nu}_1 \hat{J}_{1L} + \hat{\nu}_2 \hat{J}_{2L} + \hat{\nu}_3 \hat{J}_{3L})\,, \tag{3.9}$$
$$\hat{N}_R = \hat{N}_L\,. \tag{3.10}$$

We recall the notation $\hat{\nu}_i = \nu_i - [\nu_i]$, $\hat{J}_{iR} = J_{iR} - \frac{1}{2}$, $\hat{J}_{iL} = J_{iL} + \frac{1}{2}$, so that $\hat{J}_i = \hat{J}_{iL} + \hat{J}_{iR} = J_i$. The same remarks given at the end of section 2 apply to this model.

3.2 Three independent deformations

Here we consider three independent deformations b_i, which are the analog to the 3-parameter deformation of $AdS_5 \times S_5$ studied in [15, 23]. This model is obtained by a sequence of transformations, $(\mathrm{TsT})_{b_1}(\mathrm{TsT})_{b_2}(\mathrm{TsT})_{b_3}$. Following the same procedure as in the previous sections, we now find the spectrum

$$\alpha' M^2 = 2\Big(\hat{N}_R + \hat{N}_L - \hat{\nu}_1(\hat{J}_{1R} - \hat{J}_{1L}) - \hat{\nu}_2(\hat{J}_{2R} - \hat{J}_{2L}) - \hat{\nu}_3(\hat{J}_{3R} - \hat{J}_{3L})\Big)\,, \tag{3.11}$$
$$\hat{N}_R = \hat{N}_L\,, \tag{3.12}$$

where $\hat{\nu}_i = \nu_i - [\nu_i]$,

$$\nu_1 = \alpha'(b_3 \hat{J}_2 - b_2 \hat{J}_3)\,, \qquad \nu_2 = \alpha'(b_1 \hat{J}_3 - b_3 \hat{J}_1)\,, \qquad \nu_3 = \alpha'(b_2 \hat{J}_1 - b_1 \hat{J}_2)\,, \tag{3.13}$$

or

$$\nu_i = \alpha' \epsilon_{ijk} b_k \hat{J}_j\,. \tag{3.14}$$

In the case $b_1 = b_2 = b_3 = b$, we recover the mass spectrum (3.9) of the model of section 3.1.

For generic values of b_1, b_2, b_3, all supersymmetries are broken. An important question is whether the mass spectrum contains tachyons. To look for a tachyon, we shall consider a state with $\hat{\nu}_1$, $\hat{\nu}_2$, $\hat{\nu}_3$ different from zero, and with maximum value of $(\hat{J}_{1R} - \hat{J}_{1L})$.

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The different situations that typically arise are illustrated below by considering different regions of the parameter space.

1. All ν_i are in the interval $0 < \nu_i < 1$:

   The $S_{iR},\ S_{iL}$ satisfy the bounds

$$|S_{1R} \pm S_{2R} \pm S_{3R}| \leq \hat{N}_R + 1\,, \qquad |S_{1L} \pm S_{2L} \pm S_{3L}| \leq \hat{N}_L + 1\,. \tag{3.15}$$

   So we choose $S_{1R} = N + 1$, $S_{1L} = -N - 1$, where $N \equiv \hat{N}_R = \hat{N}_L$, $l_{1L} = l_{1R} = 0$ and, in addition,

$$l_{2R} = 1\,, \qquad l_{2L} = S_{2R} = S_{2L} = 0\,, \tag{3.16}$$
$$l_{3L} = 1\,, \qquad l_{3R} = S_{3R} = S_{3L} = 0\,. \tag{3.17}$$

   Hence we have (see (2.27))

$$\hat{J}_{1R} = N + \frac{1}{2}\,, \qquad \hat{J}_{1L} = -N - \frac{1}{2}\,, \tag{3.18}$$

$$\hat{J}_{2R} = -\frac{3}{2}\,, \qquad \hat{J}_{2L} = \frac{1}{2}\,, \qquad \hat{J}_{3R} = -\frac{1}{2}\,, \qquad \hat{J}_{3L} = \frac{3}{2}\,. \tag{3.19}$$

   For the deformation parameters, we assume $-1 < b_2 + b_3 < 0$, $0 < b_1 < 1$. Note that this range already excludes the supersymmetric case $b_1 = b_2 = b_3$. In what follows we set $\alpha' = 1$. From eq. (3.13), we find

$$\hat{\nu}_1 = |b_2 + b_3|\,, \qquad \hat{\nu}_2 = b_1\,, \qquad \hat{\nu}_3 = b_1\,. \tag{3.20}$$

   Then the mass formula takes the form

$$M^2 = 2\Big(2N + 4b_1 - |b_2 + b_3|\ (2N + 1)\Big)\,. \tag{3.21}$$

   These states become tachyonic for

$$|b_2 + b_3| > b_{\rm cr}\,, \qquad b_{\rm cr} \equiv \frac{2N + 4b_1}{2N + 1}\,. \tag{3.22}$$

   Thus, for any $N = 0, 1, 2, \ldots$, the states are tachyonic for sufficiently large $|b_2 + b_3|$. The assumption $|b_2 + b_3| < 1$ in the tachyonic regime (3.22) is satisfied by choosing $4b_1 < 1$. This leaves a wide range of b-parameters where these tachyons exist.

   Finally, one can show that fermions have positive mass squared, as expected.

2. ν_1 is in the interval $-1 < \nu_1 < 0$:

   As pointed out in section 2, in this case the GSO projection is the reversed one. Now $\hat{N}_{L,R}$ take the values $\hat{N}_{L,R} = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots$ and we have the bound

$$|S_{1R} \pm S_{2R} \pm S_{3R}| \leq \hat{N}_R + \frac{1}{2}\,, \qquad |S_{1L} \pm S_{2L} \pm S_{3L}| \leq \hat{N}_L + \frac{1}{2}\,. \tag{3.23}$$

We choose $S_{1R} = N + \frac{1}{2}$, $S_{1L} = -N - \frac{1}{2}$, $N \equiv \hat{N}_R = \hat{N}_L$, $l_{1L} = l_{1R} = 0$, so that

$$\hat{J}_{1R} = N \,, \qquad \hat{J}_{1L} = -N \,. \tag{3.24}$$

In the planes 2 and 3, we choose the same quantum numbers as in eqs. (3.16), (3.17), (3.19).

For the deformation parameters, we now assume $0 < b_2 + b_3 < 1$, $0 < b_1 < 1$. From (3.13), we now find

$$\hat{\nu}_1 = 1 - b_2 - b_3 \,, \qquad \hat{\nu}_2 = b_1 \,, \qquad \hat{\nu}_3 = b_1 \,. \tag{3.25}$$

Then the mass formula (3.11) becomes

$$M^2 = 2\Big(2N + 4b_1 - (1 - b_2 - b_3)2N\Big) = 2\Big(4b_1 + (b_2 + b_3)2N\Big) \,. \tag{3.26}$$

The state $N = -1/2$ is tachyonic for

$$b_2 + b_3 > 4b_1 \,. \tag{3.27}$$

In the supersymmetric case, $b_1 = b_2 = b_3$, the condition (3.27) is not satisfied and the state has a positive squared mass.

3. ν_1 is in the interval $-2 < \nu_1 < -1$:

   Here we have again the standard GSO projection. We choose exactly the same quantum numbers as in the case 1). But now we assume $1 < b_2 + b_3 < 2$, $0 < b_1 < 1$. The supersymmetric case $b_1 = b_2 = b_3$ is included in the discussion, and it is interesting to see how tachyons disappear. We have

$$\hat{\nu}_1 = 2 - b_2 - b_3 \,, \qquad \hat{\nu}_2 = b_1 \,, \qquad \hat{\nu}_3 = b_1 \,. \tag{3.28}$$

   Then the mass formula takes the form

$$M^2 = 2\Big(2N + 4b_1 - (2 - b_2 - b_3)(2N + 1)\Big) \,. \tag{3.29}$$

   These states are tachyonic for

$$2 - b_2 - b_3 > \frac{2N + 4b_1}{2N + 1} \,, \qquad 4b_1 < 1 \,. \tag{3.30}$$

   Now let us specialize to the supersymmetric case $b_1 = b_2 = b_3$. Then $4b_1 = 2(b_2 + b_3) = 4 - 2\hat{\nu}_1$. Hence

$$M^2 = 2\Big(2N + (4 - 2\hat{\nu}_1) - \hat{\nu}_1(2N + 1)\Big) = 2\Big((2N + 4)(1 - \hat{\nu}_1) + \hat{\nu}_1\Big) \,, \tag{3.31}$$

   which is positive definite, since, by definition, $0 < \hat{\nu}_1 < 1$.

4. Energies of short strings in the Lunin-Maldacena deformation of $AdS_5 \times S^5$

Given the parallel between the models considered here and the Lunin-Maldacena deformation $(AdS_5 \times S^5)_\beta$, an interesting question is whether there may be a relation between the corresponding string spectra. For strings of size much less than the AdS_5 or S^5 radius R, the string dynamics is essentially as in flat spacetime. We expect that the spectrum of such short strings in the Lunin-Maldacena background $(AdS_5 \times S^5)_\beta$ will have a similar structure as the spectra discussed in this paper, with the change $\alpha' \to \alpha'/\sqrt{\lambda}$, and $\alpha' b \to \hat{\beta}/\sqrt{\lambda}$, $\hat{\beta} \equiv \beta\sqrt{\lambda}$. The parameter β is assumed to be real and it is what appears in the Yang-Mills superpotential $\mathrm{Tr}[e^{i\pi\beta}\Phi_1\Phi_2\Phi_3 - e^{-i\pi\beta}\Phi_1\Phi_3\Phi_2]$. Taking a flat-space limit of the string spectrum in $(AdS_5 \times S^5)_\beta$ necessarily requires sitting at some point of S^5 (e.g. $r_3 = 1$, $r_1 = r_2 = 0$), and considering short strings. This procedure breaks the Z_3 symmetry associated with exchange of the 1-2-3 planes. The resulting string spectrum approaches a truncation of the spectrum of the model of section 3.1 (the spectrum of short strings in $(AdS_5 \times S^5)_\beta$ cannot be described by the full spectrum (3.9), because the latter involves oscillator modes associated with six dimensions, as opposed to the five dimensions of S^5). One interesting problem would be to compare the string spectrum (3.9) with the energy of semiclassical short strings in $(AdS_5 \times S^5)_\beta$ having $1 \ll \hat{J}_i \ll \sqrt{\lambda}$.

The existence of tachyons in the three-parameter model of section 3.2 suggests that there could also be tachyons in the analog model of [15, 23]. It will be difficult to see such possible tachyons in a semiclassical approximation of large N. In particular, note that for the existence of the above tachyon states, it is essential that there is a "1" in $2N + 1$ in (3.21) and in (3.29). The origin of this 1 is a normal ordering contribution, and it is negligible in a semiclassical approximation where $N \gg 1$. There are tachyons at any given string level number N in some regions of parameter space, including low values of N, in particular $N = 0$. From the point of view of the dual gauge theory, low N means short operators (the string level number N should not be confused of course with N of U(N)). It would be interesting to see if there is a counterpart of the tachyon instabilities in the dual non-supersymmetric gauge theory of [15, 23]. It would also be interesting to understand the limit taken at the end of section 2 (related to $\beta \to p/q$) within the $\mathcal{N} = 1$ superconformal gauge theory.

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